

DECOMPOSABILITY OF QUOTIENTS BY COMPLEX CONJUGATION FOR RATIONAL AND ENRIQUES SURFACES

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ABSTRACT. The quotients $Y = X/\text{conj}$ by the complex conjugation $\text{conj}: X \rightarrow X$ for complex rational and Enriques surfaces X defined over \mathbb{R} are shown to be diffeomorphic to connected sums of $\overline{\mathbb{CP}}^2$, whenever Y are simply connected.

§1 INTRODUCTION

1.1. The main results. Given a complex algebraic surface X defined over \mathbb{R} , one can consider *the complex conjugation*, $\text{conj}: X \rightarrow X$, and the quotient $Y = X/\text{conj}$. If X is nonsingular, then Y is a closed 4-manifold, which inherits an orientation and smooth structure from X , so that the quotient map, $q: X \rightarrow Y$, is an orientation preserving and smooth double covering branched along the fixed points set, $X_{\mathbb{R}}$, of the conjugation. We call the latter *the real part of X* .

The main results of this paper are the following theorems.

1.1.1. Theorem. *If X is a rational nonsingular complex surface defined over \mathbb{R} with $X_{\mathbb{R}} \neq \emptyset$, then Y is diffeomorphic to $\#_m \overline{\mathbb{CP}}^2$, where $m = \frac{1}{2}(\chi(X) + \chi(X_{\mathbb{R}})) - 2$.*

1.1.2. Theorem. *If X is an Enriques surface defined over \mathbb{R} and Y is simply connected, then Y is diffeomorphic to $\#_m \overline{\mathbb{CP}}^2$, where $m = \frac{1}{2}\chi(X_{\mathbb{R}}) + 4$.*

By making use of the relation between the Euler characteristics and the signatures of X and Y one gets formulas, $b_2^+(Y) = p_g$ and $b_2^-(Y) = \frac{1}{2}(\chi(X) + \chi(X_{\mathbb{R}})) - 2$, for any algebraic surface X of genus p_g provided $b_1(X) = 0$. Therefore, Freedman's and Donaldson's theorems imply that Y is *homeomorphic* to $\#_m \overline{\mathbb{CP}}^2$ if Y is simply connected and X has genus 0. Further, Y is simply connected if X is simply connected and $X_{\mathbb{R}} \neq \emptyset$. It can also be simply connected even when X is not, and the most topological types of real Enriques surfaces provide such examples. The goal of the above theorems is,

therefore, to show that the quotients Y for rational and Enriques surfaces X cannot be exotic $\#_m \overline{\mathbb{CP}}^2$.

1.2. History of the subject. In the case $X = \mathbb{CP}^2$ the diffeomorphism $Y \cong S^4$ is known as the Kuiper–Massey theorem [K,M]. For a real quadric or a cubic surface, $X \subset \mathbb{CP}^3$, the diffeomorphism type of Y was determined in [L]. In [F1,F2] the author found the differential type of Y for several families of real algebraic surfaces X and, in particular, for certain rational surfaces. The Kuiper–Massey theorem implies also that a blow-up at a point $P \in X_{\mathbb{R}}$ preserves the quotient, $(X \# \overline{\mathbb{CP}}^2)/\text{conj} \cong Y \# S^4 \cong Y$. Blow-up at a pair of conjugated imaginary points of X descend, obviously, to a blow-up of the quotient Y . It follows that it suffices to prove Theorem 1.1.1 only for real minimal models of rational surfaces.

1.3. Structure of the paper. In §2, we recall some basic definitions related to real algebraic surfaces and formulate the classification theorem for real minimal models of rational surfaces. In §3 we analyze certain classes of real minimal rational surfaces and complete the proof of Theorem 1.1.1. In §4 we deduce Theorem 1.1.2 from Theorem 1.1.1.

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§2 REAL MINIMAL MODELS FOR RATIONAL SURFACES

2.1. Real surfaces. By a real variety (a real surface, a real curve) we mean a pair, (X, conj) , where X is a complex variety and $\text{conj}: X \rightarrow X$ an anti-holomorphic involution, called *the complex conjugation* or *the real structure*. A morphism, $X_1 \rightarrow X_2$, between real varieties, (X_i, conj_i) , $i = 1, 2$, is called *real* if it commutes with the complex conjugations in X_1 and X_2 . It is not difficult to see that if X is an algebraic complex variety, then there exists a real embedding $X \subset \mathbb{CP}^N$, which makes X an algebraic variety over \mathbb{R} (such an embedding is defined by $\mathcal{L} \otimes \text{conj}^*(\mathcal{L})$, where \mathcal{L} is a very ample linear bundle on X , cf. [CD]).

By a *real rational surface* we mean a real surface which is rational as a complex surface. A ruled surface $f: X \rightarrow B$ is called *real* if X and B are supplied with a real structure so that the ruling f is a real morphism. By a real rational conic bundle we mean a real morphism $f: X \rightarrow \mathbb{CP}^1$, which generic fibers are rational curves and singular fibers split into wedges of two rational curves.

2.2. Real minimal models. A blow-up $X' \rightarrow X$ at a real point of a real surface (X, conj) will be called *an elementary real blow-up of type 1*. A blow-up at a pair of conjugated imaginary points of X will be called *an elementary real blow-up of type 2*. It is easy to see that in the both cases X' inherits a real structure making $X' \rightarrow X$ a real morphism. It is well known (cf. [S]) that any real birational equivalence between real surfaces can be decomposed into a sequence of elementary real blow-ups and downs. A real surface (X, conj) is called *real minimal* if every real birational morphism $X \rightarrow X'$, where X' is a smooth real surface, is an isomorphism. In the other words, X is minimal if for any exceptional curve, $C \subset X$, (i.e. rational curve with $C \circ C = -1$), we have $C \circ \text{conj}(C) \geq 1$.

As it is mentioned above, a real blow-up of type 1 does not change the quotient Y , a real blow-up of type 2 descends to a blow-up on Y and the proof of Theorem 1.1.1 will be completed after we check its statement for minimal models of real rational surfaces.

According to Comessatti [C], the minimal models of real rational surfaces are classified as follows (see also [CP], [S]).

2.2.1. Theorem. *Assume that X is a minimal real rational surface. Then X is real isomorphic to one of the following types.*

- (1) \mathbb{CP}^2 with the usual real structure;
- (2) a quadric in \mathbb{CP}^3 which does not contain real points ($X_{\mathbb{R}} = \emptyset$);
- (3) a quadric with $X_{\mathbb{R}} \cong S^2$;
- (4) a real ruled rational surface, $X \cong F_n$, $n \geq 0$, which has the real part $X_{\mathbb{R}}$ homeomorphic to a Klein bottle if n is odd, and either homeomorphic to a torus or empty if n is even;
- (5) a real conic bundle over \mathbb{CP}^1 with an even number, $2n$, of singular fibers, which are all real and consist of pairs of complex conjugated exceptional curves; in this case $X_{\mathbb{R}} \cong nS^2$;
- (6) a real del Pezzo surface of degree 2 with $X_{\mathbb{R}} \cong 4S^2$, which can be constructed as a double plane with the branch locus a real quartic;
- (7) a real del Pezzo surface of degree 1 with $X_{\mathbb{R}} \cong \mathbb{RP}^2 \amalg 4S^2$, which can be constructed as a real double quadratic cone, $X \rightarrow Q$, whose branch locus is the intersections of the cone $Q \subset \mathbb{CP}^3$, with a cubic surface, and a further branch point is the vertex of Q .

Here \amalg stands for disjoint union and nS^2 for disjoint union of n spheres.

Note that to prove Theorem 1.1.1 we need to consider only the cases (4), (5) and (7), since in the case (2) $X_{\mathbb{R}} = \emptyset$ and in the cases (1), (3), (6) the statement of Theorem 1.1.1 is well known (see [K,M] for (1), [L] for (3) and [F1] for (6)). The cases (4), (5), (6) are considered in the next section.

§3 PROOF OF THEOREM 1.1.1

3.1. Real ruled surfaces. Consider a real ruled rational surface $p: X \rightarrow \mathbb{CP}^1$. Factorization by conj gives a smooth map $\hat{p}: Y \rightarrow \Delta$, where $\Delta = \mathbb{CP}^1 / \text{conj}$, such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{p} & \mathbb{CP}^1 \\ q \downarrow & & \downarrow \\ Y & \xrightarrow{\hat{p}} & \Delta. \end{array}$$

Split Δ into a union of a smaller disc, $\Delta_1 \subset \Delta$, and the annulus, $\Sigma = \text{Cl}(\Delta - \Delta_1)$. It can be easily seen that \hat{p} is a smooth fibering over the interior of Δ with a fiber S^2 , therefore $\hat{p}^{-1}(\Delta_1) \cong D^2 \times S^2$. Moreover, $\hat{p}^{-1}(\Sigma) \cong S^1 \times D^3$. This is because the product of \hat{p} and the regular smooth retraction, $\Sigma \rightarrow S^1$, is a smooth D^3 -fibering, $r: \hat{p}^{-1}(\Sigma) \rightarrow S^1$. Further, it is well known that a 4-manifold with the boundary $S^1 \times S^2$ can be filled up by $S^1 \times D^3$ in a unique way, thus, $Y \cong S^4$.

3.2. Conic bundles. In this section we prove the following

3.2.1. Proposition. *If X is a real minimal conic bundle with m singular fibers, then Y is diffeomorphic to $\#_m \overline{\mathbb{CP}}^2$.*

Proof. Let $p: X \rightarrow \mathbb{CP}^1$ be a real minimal conic bundle and $\hat{p}: Y \rightarrow \Delta$ obtained by factorization by conj , as in 3.1. Let $p^{-1}(x_i) = E'_i \cup E''_i$, $i = 1, \dots, m$, be the singular fibers of X , $E_i = p^{-1}(x_i) / \text{conj} = \hat{p}^{-1}(x_i)$, $E'_i \cap E''_i = \{P_i\}$, $i = 1, \dots, m$, and x_1, \dots, x_m are ordered consecutively on the circle $\partial\Delta$. We call $x \in \mathbb{RP}^1 \subset \mathbb{CP}^1$ a point of type 1 if the restriction of conj to $p^{-1}(x)$ has fixed points (hence, $\hat{p}^{-1}(x) = p^{-1}(x) / \text{conj} \cong D^2$), and a point of type 2 if conj it has no (then, $\hat{p}^{-1}(x) \cong \mathbb{RP}^2$). It is well known (and easy to check) that $\hat{p}|_{X_{\mathbb{R}}}$ is a smooth map, which has critical points only at P_i , $i = 1, \dots, m$, and that the type of a point $x \in \partial\Delta$ changes as we cross x_i (cf. [S]), which explains why m is even, $m = 2n$.

Let l_i , $i = 1, \dots, m-1$, denote the closed arcs in $\partial\Delta$ between x_i and x_{i+1} and l_m the arc between x_m and x_1 ; all arcs are chosen not to contain in their interior the points x_i . Further, let us choose the order of x_i so that l_i consists of points of type 1 for odd i and type 2 for even i . Consider small regular closed disjoint neighborhoods $N_i \subset \Delta$ of the arcs l_{2i} , $i = 1, \dots, n$ (see Figure 1), and put $N_0 = \text{Cl}(\Delta - \cup_{i=1}^n N_i)$, $A_i = \hat{p}^{-1}(N_i)$, $i = 0, \dots, n$.

3.2.2 Lemma. *A_0 is diffeomorphic to $S^4 - n\overset{\circ}{D}^4$ (a 4-sphere with n disjoint open regular 4-discs removed).*

Proof of Lemma 3.2.2. Let T_i be small disjoint closed regular neighborhoods of $l_{2i-1} \cap N_0$ in N_0 , $\Delta_1 = \text{Cl}(N_0 - \cup_{i=1}^n T_i)$. Then $\hat{p}^{-1}(T_i) \cong I \times D^3$ are 3-handles attached to $\hat{p}^{-1}(\Delta_1) \cong D^2 \times S^2$, with the cores $\text{pt} \times S^2 \subset D^2 \times S^2$ (see Figure 1). Such a surgery yields $S^4 - n\mathring{D}^4$. \square

3.2.3. Lemma. $A_i, i = 1, \dots, n$, is diffeomorphic to $\overline{\mathbb{CP}}^2 \# \overline{\mathbb{CP}}^2 - \mathring{D}^4$.

Proof of Lemma 3.2.3. Let $S_i \subset \Delta$ be a small closed regular neighborhood of $x_i, i = 1, \dots, 2n$ and $B_i = \hat{p}^{-1}(S_i)$. Denote by H the total space of a smooth 2-disc fiber bundle over S^2 with the normal number -2 . Note, first, that B_i is a regular neighborhood of E_i and that $E_i \cong S^2$ has self-intersection -2 , which implies that $B_i \cong H$. A_i is obtained by gluing together 2 copies of H (B_{2i} and B_{2i+1}) along the part of their boundary $\mathbb{RP}^3 - \mathring{D}^4 \subset \mathbb{RP}^3 = \mathring{H}$. We can extend the gluing map $\hat{f}: \mathbb{RP}^3 - \mathring{D}^4 \rightarrow \mathbb{RP}^3 - \mathring{D}^4$ to $f: \mathbb{RP}^3 \rightarrow \mathbb{RP}^3$ and get a closed manifold $R = H \cup_f H$. Further, obviously, $A_i \cong R - \mathring{D}^4$ and since $\text{Diff}_+(\mathbb{RP}^3)$ is known to be connected (cf. [H]), Q is diffeomorphic to a standard pattern, obtained by gluing of two copies of H along an orientation reversing diffeomorphism $g: \mathbb{RP}^3 \rightarrow \mathbb{RP}^3$ (g is orientation reversing since $b_2^+(A_i) = 0$). It is a standard and trivial exercise in Kirby calculus to check that if g is a mirror reflection, then $H \cup_g H \cong \overline{\mathbb{CP}}^2 \# \overline{\mathbb{CP}}^2$. \square

FIGURE 1. Proof of Proposition 3.2.1

By the above lemmas, Y is obtained from $A_0 \cong S^4 - n\mathring{D}^4$ by filling the “holes” with $A_i \cong (\overline{\mathbb{CP}}^2 \# \overline{\mathbb{CP}}^2) - \mathring{D}^4$, which yields $\#_m \overline{\mathbb{CP}}^2$. \square

3.3. Del Pezzo surfaces of degree 1. Figure 2 shows a construction of a curve $A = Q \cap C$, the intersection of a quadratic cone Q with a cubic surface $C \subset \mathbb{CP}^3$, which real part $A_{\mathbb{R}}$ consists of 4 ovals, i.e. components contractible in $Q_{\mathbb{R}} - \{P\}$, and a component non-contractible in $Q_{\mathbb{R}} - \{P\}$. Consider

a double covering $p: X \rightarrow Q$ branched along A and at the vertex of Q , and choose the one of two possible liftings to X of the real structures in Q for which p maps $X_{\mathbb{R}}$ into the domain, which consists of 4 discs bounded by the ovals of $A_{\mathbb{R}}$ and of the part of $Q_{\mathbb{R}}$ bounded by the non-contractible component and P , as it is shown on Figure 2 (standard details on liftings of real structures the reader can find, e.g., in [F1]).

FIGURE 2. The quadratic cone Q is the double plane branched along a pair of real lines, L_1, L_2 . A cubic surface C is the pull-back of the cubic curve obtained by a perturbation of 3 lines (left figure). The real components of $X_{\mathbb{R}}$ are projected onto the shaded domains (right figure).

From the construction of A it can be easily seen that a pair of ovals of $A_{\mathbb{R}}$ can be fused by a deformation of C , after passing a nodal singularity on $A_{\mathbb{R}}$. This gives a deformation of double coverings fusing a pair of real components of $X_{\mathbb{R}}$. It is shown in [L] that the effect for the quotient $Y = X/\text{conj}$ of such a deformation is a blow-down (see also [F1], [W]). Since Theorem 1.1.1 is already set up for rational surfaces with 4 real components, we have proved it in our case as well. \square

§4 PROOF OF THEOREM 1.1.2

Let $p: \tilde{X} \rightarrow X$ be the double covering of a real Enriques surface (X, conj) by K3 surface \tilde{X} . Denote by $c_i: \tilde{X} \rightarrow \tilde{X}$, $i = 1, 2$, the anti-holomorphic involutions, covering conj ; c_1 and c_2 commute and give in product the covering transformation $t: \tilde{X} \rightarrow \tilde{X}$ of p (cf. [DK]). We assume that one of the involutions, say c_1 , has a fixed point, since, otherwise $\mathbb{Z}/2 \times \mathbb{Z}/2$ acts freely on \tilde{X}

and $Y = X/\text{conj}$ is not simply connected. Further, c_2 (or, equally, t) induces an involution $c'_2: X' \rightarrow X'$ on the quotient space $X' = X/c_1$.

Now, one can change the complex structure on \tilde{X} so that c_1 becomes a holomorphic involution, whereas c_2 and, therefore, t anti-holomorphic. To get it we pick up a conj-symmetric Kalabi-Yau (Ricci flat) metric on \tilde{X} and following the idea of Donaldson [D] use that K3 surfaces are hyper Kähler to vary the complex structure on X . The only novelty related to Enriques surfaces is that we need to choose the metric on X to be symmetric with respect to both c_2 and t . Such a metric is the pull back of a conjugation symmetric metric on X (more detailed explanation can be found in [DK2]).

Quotients of K3 surfaces by holomorphic involutions, which have non-empty fixed point set, are known to be rational (see, e.g., [W]), therefore, X' gets a structure of rational surface with the real structure c'_2 . Hence, by Theorem 1.1.1, $Y = X'/c'_2$ splits into a connected sum of $\overline{\mathbb{CP}}^2$'s provided it is simply connected. \square

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